RELATIONS OF AL FUNCTIONS OVER SUBVARIETIES IN A HYPERELLIPTIC JACOBIAN

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Abstract. The sine-Gordon equation has hyperelliptic al function solutions over a hyperelliptic Jacobian for $y^2 = f(x)$ of arbitrary genus g. This article gives an extension of the sine-Gordon equation to that over subvarieties of the hyperelliptic Jacobian. We also obtain the condition that the sine-Gordon equation in a proper subvariety of the Jacobian is defined.

1. Introduction

For a hyperelliptic curve C_g given by an affine curve $y^2 = \prod_{i=1}^{2g+1}(x - b_i)$, where b_i's are complex numbers, we have a Jacobian $\mathcal{J}_g$ as a complex torus $\mathbb{C}^g/\Lambda$ by the Abel map ω [Mu]. Due to the Abelian theorem, we have a natural morphism from the symmetrical product $\mathrm{Sym}^g(C_g)$ to the Jacobian $\mathcal{J}_g \approx \omega[\mathrm{Sym}^g(C_g)]/\Lambda$. As zeros of an appropriate shifted Riemann theta function over $\mathcal{J}_g$, the theta divisor is defined as

$$\Theta := \omega[\mathrm{Sym}^{g-1}(C_g)]/\Lambda$$

which is a subvariety of $\mathcal{J}_g$. Similarly, it is natural to introduce a subvariety

$$\Theta_k := \omega[\mathrm{Sym}^k(C_g)]/\Lambda$$

and a sequence,

$$\Theta_0 \subset \Theta_1 \subset \Theta_2 \subset \cdots \subset \Theta_{g-1} \subset \Theta_g \equiv \mathcal{J}_g$$

Vanhaecke studied the structure of these subvarieties as stratifications of the Jacobian $\mathcal{J}_g$ using the strategies developed in the studies of the infinite dimensional integrable system [V1]. He showed that these stratifications of the Jacobian are connected with stratifications of the Sato Grassmannian. Further Vanhaecke investigated Lie-Poisson structures in the Jacobian in [V2]. He showed that invariant manifolds associated with Poisson brackets can be identified with these strata; it implies that the strata are characterized by the Lie-Poisson structures. He also showed that the Poisson brackets are connected with a finite-dimensional integrable system, Henon-Heiles system. Following the study, Abenda and Fedorov [AF] investigated these strata and their relations to Henon-Heiles system and Neumann systems.

On the other hand, functions over the embedded hyperelliptic curve Θ_1 in a hyperelliptic Jacobian $\mathcal{J}_g$ were also studied from viewpoint of number theory in [C, G, Ô]. In [Ô], Ônishi also investigated the sequence of the subvarieties, and explicitly studied behaviors of functions over them in order to obtain higher genus

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analog of the Frobenius-Stickelberger relations for genus one case. Though Vanhaecke, Abenda and Fedorov found some relations of functions over these subvarieties explicitly using the infinite universal grassmannians and so-called Mumford's UVW expressions [Mu], Ônishi gave more explicit relations on some functions over the subvarieties using the theory of hyperelliptic functions in the nineteenth century fashion [Ba1, Ba2, Ba3].

In this article, we will also investigate some relations of functions over the subvarieties based upon the studies of the hyperelliptic function theory developed in the nineteenth century [Ba2, Ba3, W]. Especially this article deals with the "sine-Gordon equation" over there.

Modern expressions of the sine-Gordon equation in terms of Riemann theta functions were given in [[Mu] 3.241],

$$\frac{\partial}{\partial t_{\mathrm{P}}}\frac{\partial}{\partial t_{\mathrm{Q}}}\log([2\mathrm{P}-2\mathrm{Q}]) = A([2\mathrm{P}-2\mathrm{Q}]-[2\mathrm{Q}-2\mathrm{P}]), \tag{1.1}$$

where P and Q are ramified points of C_g, A is a constant number, $[D]$ is a meromorphic function over $\mathrm{Sym}^g(C_g)$ with a divisor D for each C_g and $t_{\mathrm{P}'}$ is a coordinate in the Jacobi variety such that it is identified with a local parameter at a ramified point P' up to constant.

In the previous work [Ma], we also studied (1.1) using the fashion of the nineteenth century. In [W] Weierstrass defined al function by $\mathrm{al}_r := \gamma_r\sqrt{F_g(b_r)}$ and $F_g(z) := (x_1 - z)\cdots(x_g - z)$ over $\mathcal{J}_g$ with a constant factor γ_r. Let $\gamma_r = 1$ in this article. As Weierstrass implicitly seemed to deal with it, (1.1) is naturally described by al-functions as [Ma],

$$\frac{\partial^2}{\partial v_1^{(g)}\partial v_2^{(g)}}\log\frac{\mathrm{al}_r}{\mathrm{al}_s} = \frac{1}{(b_r-b_s)}\left(f'(b_s)\left(\frac{\mathrm{al}_r}{\mathrm{al}_s}\right)^2 + f'(b_r)\left(\frac{\mathrm{al}_s}{\mathrm{al}_r}\right)^2\right). \tag{1.2}$$

Here $f'(x) := df(x)/dx$ and $v^{(g)}$'s are defined in (2.4). ((1.2) was obtained in the previous article [Ma] by more direct computations and will be shown as Corollary 3.3 in this article). We call (1.2) Weierstrass relation in this article.

In this article, we will introduce an "al" function over the subvariety in the Jacobian, $\mathrm{al}_r^{(m)} := \sqrt{F_m(b_r)}$ and $F_m(z) := (x_1-z)\cdots(x_m-z)$ for a point $((x_1,y_1),\cdots,(x_m,y_m))$ in the symmetric product of the m curves $\mathrm{Sym}^m C_g$ $(m = 1,\cdots,g-1)$. In [Mu], Mumford dealt with F_m function (he denoted it by U) for $1 \le m < g$ and studied the properties. Further Abenda and Fedorov also studied some properties of the $\mathrm{al}_r^{(m)}$ and F_m functions in [AF] though they did not mention about Weierstrass's paper nor the relation (1.2). We will consider a variant of the Weierstrass relation (1.2) to $\mathrm{al}_r^{(m)}$ over subvariety in non-degenerated and degenerated hyperelliptic Jacobian.

As in our main theorem 3.1, even on the subvarieties, we have a similar relation to (1.1),

(1.3)

$$\frac{\partial}{\partial v_r^{(m)}}\frac{\partial}{\partial v_s^{(m)}}\log\frac{\mathrm{al}_r^{(m)}}{\mathrm{al}_s^{(m)}} = \frac{1}{(b_r-b_s)}\left(\frac{f'(b_r)}{(Q_m^{(2)}(b_r))^2}\left(\frac{\mathrm{al}_s^{(m)}}{\mathrm{al}_r^{(m)}}\right)^2 + \frac{f'(b_s)}{(Q_m^{(2)}(b_s))^2}\left(\frac{\mathrm{al}_r^{(m)}}{\mathrm{al}_s^{(m)}}\right)^2\right)$$
$$+ \text{reminder terms}.$$

Here $Q_m^{(2)}$ is defined in (2.2). We regard (1.3) or (3.1) as a subvariety version of the Weierstrass relation (1.2). In fact, (1.3) contains the same form as (1.1) up to the factors $(Q_m^{(2)}(b_t))^2$ $(t = r, s)$ and the reminder terms. Thus (1.3) or (3.1) should be regarded as an extension of the sine-Gordon equation (1.2) over the Jacobian to that over the subvariety of the Jacobian.

Further a certain degenerate curve, the remainders in (1.3) vanishes. Then we have a relations over subvarieties in the Jacobian, which is formally the same as the Weierstrass relations (1.2) up to the factors $(Q_m^{(2)}(b_t))^2$ $(t = r, s)$, which means that we can find solutions of sine-Gordon equation over subvarieties in hyperelliptic Jacobian. We expect that our results shed a light on the new field of a relation between "integrability" and a subvariety in the Jacobian, which was brought off by [V1, V2, AF].

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2. Differentials of a Hyperelliptic Curve

In this section, we will give our conventions of hyperelliptic functions of a hyperelliptic curve C_g of genus g $(g > 0)$ given by an affine equation,

$$\begin{aligned} y^2 = f(x) &= (x - b_1)(x - b_2) \cdots (x - b_{2g})(x - b_{2g+1}) \\ &= Q(x)P(x), \end{aligned} \tag{2.1}$$

where b_j's are complex numbers. Here we use the expressions $Q(x) := Q_m^{(1)}(x)Q_m^{(2)}(x)$,

$$\begin{aligned} Q_m^{(1)}(x) &:= (x - a_1)(x - a_2) \cdots (x - a_m), \\ Q_m^{(2)}(x) &:= (x - a_{m+1})(x - a_{m+2}) \cdots (x - a_g), \\ P(x) &:= (x - c_1)(x - c_2) \cdots (x - c_g)(x - c), \end{aligned} \tag{2.2}$$

where $a_k \equiv b_k$, $c_k \equiv b_{g+k}$, $(k = 1, \cdots, g)$ $c \equiv b_{2g+1}$.

Definition 2.1. [Ba1, Ba2]

For a point $(x_i, y_i) \in C_g$, we define the following quantities.

(1) The unnormalized differentials of the first kind are defined by,

$$dv_k^{(g,i)} := \frac{Q(x_i)dx_i}{2(x_i - a_k)Q'(a_k)y_i}, \quad (k = 1, \cdots, g) \tag{2.3}$$

(2) The Abel map for g-th symmetric product of the curve C_g is defined by,

$$v^{(g)} \equiv (v_1^{(g)}, \cdots, v_g^{(g)}) : \mathrm{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,$$

$$\left(v_k^{(g)}((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} \int_{\infty}^{(x_i, y_i)} dv_k^{(g,i)} \right). \tag{2.4}$$

(3) For $v^{(g)} \in \mathbb{C}^g$, we define the subspace,

$$\Xi_m := v^{(g)}(\mathrm{Sym}^m(C_g) \times (a_{m+1}, 0) \times \cdots \times (a_g, 0))/\mathbf{\Lambda}. \tag{2.5}$$

Here $\mathbb{C}$ is a complex field and $\mathbf{\Lambda}$ is a g-dimensional lattice generated by the related periods or the hyperelliptic integrals of the first kind.

The Jacobi variety $\mathcal{J}_g$ are defined as complex torus as $\mathcal{J}_g := \Xi_g$. As Ξ_m $(m < g)$ is embedded in $\mathcal{J}_g$ whose complex dimension as subvariety is m, the differential forms $(dv_k^{(g)})_{k=1,\cdots,g}$ are not linearly independent. We select linearly independent bases such as $v_k^{(m)} := v_k^{(g)}((x_1, y_1), \cdots, (x_m, y_m), (a_{m+1}, 0), \cdots, (a_g, 0))$, $(k = 1, \cdots, m)$ at Ξ_m.

$$\Xi_0 \subset \Xi_1 \subset \Xi_2 \subset \cdots \subset \Xi_{g-1} \subset \Xi_g \equiv J_g$$

For $(x_1, \cdots, x_m) \in \mathrm{Sym}^m(C_g)$, we introduce

$$F_m(x) := (x - x_1) \cdots (x - x_m), \tag{2.6}$$

and in terms of $F_m(x)$, a hyperelliptic al-function over $(v^{(m)}) \in \Xi_m$, [Ba2 p.340, W],

$$\mathrm{al}_r^{(m)}(v^{(m)}) = \sqrt{F_m(b_r)}. \tag{2.7}$$

Further we introduce $m \times m$-matrices,

$$\mathcal{M}_m := \begin{pmatrix} \dfrac{1}{x_1 - a_1} & \dfrac{1}{x_2 - a_1} & \cdots & \dfrac{1}{x_m - a_1} \\ \dfrac{1}{x_1 - a_2} & \dfrac{1}{x_2 - a_2} & \cdots & \dfrac{1}{x_m - a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{x_1 - a_m} & \dfrac{1}{x_2 - a_m} & \cdots & \dfrac{1}{x_m - a_m} \end{pmatrix},$$

$$\mathcal{Q}_m = \begin{pmatrix} \sqrt{\dfrac{Q(x_1)}{P(x_1)}} & & & \\ & \sqrt{\dfrac{Q(x_2)}{P(x_2)}} & & \\ & & \ddots & \\ & & & \sqrt{\dfrac{Q(x_m)}{P(x_m)}} \end{pmatrix}, \quad \mathcal{A}_m = \begin{pmatrix} Q'(a_1) & & & \\ & Q'(a_2) & & \\ & & \ddots & \\ & & & Q'(a_m) \end{pmatrix}.$$

Lemma 2.2. (1)

$$\det \mathcal{M}_m = \frac{(-1)^{m(m-1)/2} P(x_1, \cdots, x_m) P(a_1, \cdots, a_m)}{\prod_{k,l}(x_k - a_l)},$$

where

$$P(z_1, \cdots, z_m) := \prod_{i<j} (z_i - z_j).$$

(2)

$$\mathcal{M}_m^{-1} = \left[\left(\frac{F_m(a_j) Q_m^{(1)}(x_i)}{F_m'(x_i) Q_m^{(1)\prime}(a_j)(a_j - x_i)} \right)_{i,j} \right],$$

where $F_m'(x) := dF_m(x)/dx$ *and* $Q_m^{(1)\prime}(x) = dQ_m^{(1)}(x)/dx$.

(3)

$$(2.8)\qquad (\mathcal{M}\mathcal{Q})^{-1}\mathcal{A} = \left[\left(\frac{2y_i F_m(a_j)}{F'_m(x_i)Q_m^{(2)}(x_i)(a_j - x_i)}\right)_{i,j}\right].$$

Proof. (1) is a well-known result [T]. Since the zero and singularity in the left hand side give the right hand side as

$$C\mathrm{P}(x_1, \cdots, x_m)\mathrm{P}(a_1, \cdots, a_m)/\prod_{k,l}(x_k - a_l),$$

for a certain constant C. In order to determine C, we multiply $\prod_{k,l}(x_k - a_l)$ both sides and let $x_1 = a_1$, $x_2 = a_2$, $\cdots$, and $x_m = a_m$. Then C is determined as above. (2) is obtained by the Laplace formula using the minor determinant for the inverse matrix. On (3) we note that $Q_m^{(1)}Q_m^{(2)} = Q(x)$ in (2.2) and thus $Q_m^{(1)}(x)\sqrt{P(x)/Q(x)} = y/Q_m^{(2)}$. Then we obtain (3). $\square$

Corollary 2.3. *Let $\partial_{v_i}^{(r)} := \partial/\partial v_i^{(r)}$, and $\partial_{x_i} := \partial/\partial x_i$.*

$$(2.9)\qquad \begin{pmatrix} \partial_{v_1} \\ \partial_{v_2} \\ \vdots \\ \partial_{v_m} \end{pmatrix} = 2(\mathcal{M}\mathcal{Q}_m)^{-1}\mathcal{A}_m \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_m} \end{pmatrix}.$$

3. Weierstrass relation on Ξ_m

The hyperelliptic solution of the sine-Gordon equation over $\mathcal{J}_g$ related to ramified points $(a_1, 0)$ and $(a_2, 0)$ is obtained as (1.1) by Mumford [Mu], whose expression in an old fashion is the Weierstrass relation (1.2). Let us consider an extension of the Weierstrass relation (1.2) over Ξ_m as our main theorem. We will give the theorem as follows.

Theorem 3.1. $\mathrm{al}_r^{(m)}$ *and* $\mathrm{al}_s^{(m)}$ $(r, s \in \{1, 2, \cdots, m\})$ *over* Ξ_m *in (2.5) obey the relation,*

(3.1)

$$\begin{aligned}
&\frac{\partial}{\partial v_r^{(m)}}\frac{\partial}{\partial v_s^{(m)}}\log\frac{\mathrm{al}_r^{(m)}(v^{(m)})}{\mathrm{al}_s^{(m)}(v^{(m)})} \\
&\qquad = \frac{1}{(a_r - a_s)}\left(\frac{f'(a_r)}{(Q_m^{(2)}(a_r))^2}\left(\frac{\mathrm{al}_s^{(m)}(v^{(m)})}{\mathrm{al}_r^{(m)}(v^{(m)})}\right)^2 + \frac{f'(a_s)}{(Q_m^{(2)}(a_s))^2}\left(\frac{\mathrm{al}_r^{(m)}(v^{(m)})}{\mathrm{al}_s^{(m)}(v^{(m)})}\right)^2\right) \\
&\qquad + \frac{f'(a_{m+1})(\mathrm{al}_r^{(m)}(v^{(m)}))^2(\mathrm{al}_s^{(m)}(v^{(m)}))^2(a_r - a_s)}{(a_{m+1} - a_r)(a_{m+1} - a_s)(\mathrm{al}_{m+1}^{(m)}(v^{(m)}))^4(Q_m^{(2)\prime}(a_{m+1}))^2} \\
&\qquad + \cdots\cdots \\
&\qquad + \frac{f'(a_g)(\mathrm{al}_r^{(m)}(v^{(m)}))^2(\mathrm{al}_s^{(m)}(v^{(m)}))^2(a_r - a_s)}{(a_g - a_r)(a_g - a_s)(\mathrm{al}_g^{(m)}(v^{(m)}))^4(Q_m^{(2)}(a_g)')^2}.
\end{aligned}$$

Proof. From (2.7), we will consider the following formula instead of (3.1) without loss of generality,

(3.2)
$$\frac{\partial}{\partial v_1^{(m)}}\frac{\partial}{\partial v_2^{(m)}}\log\frac{F_m(a_1)}{F_m(a_2)} = 2\frac{F_m(a_1)F_m(a_2)}{(a_1-a_2)}\Big(\frac{f'(a_1)}{F_m(a_1)^2(Q_m^{(2)}(a_1))^2}+\frac{f'(a_2)}{F_m(a_2)^2(Q_m^{(2)}(a_1))^2}$$
$$+\frac{f'(a_{m+1})(a_1-a_2)^2}{(a_{m+1}-a_1)(a_{m+1}-a_2)F_m(a_{m+1})^2(Q_m^{(2)\prime}(a_{m+1}))^2}$$
$$+\cdots$$
$$+\frac{f'(a_g)(a_1-a_2)^2}{(a_g-a_1)(a_g-a_2)F_m(a_g)^2(Q_m^{(2)\prime}(a_g))^2}\Big).$$

Before we start the proof, we will comment on our strategy, which is essentially the same as [Ba3]. First we translate the words of the Jacobian into those of the curves; we rewrite the differentials $v_{(r)}^{(m)}$'s in terms of the differentials over curves as in (3.3). We count the residue of an integration and use a combinatorial trick as in Lemma 3.2. Then we will obtain (3.2).

From (2.8) and (2.9), the derivative v's over Ξ_m in (2.5) are expressed by the affine coordinate x_i's,

$$\frac{\partial}{\partial v_i^{(m)}} = F_m(a_i)Q_m^{(2)}(a_i)\sum_{j=1}^{m}\frac{2y_j}{F'_m(x_j)Q_m^{(2)}(x_j)(x_j-a_i)}\frac{\partial}{\partial x_j}. \tag{3.3}$$

The right hand side of (3.2) becomes,

$$\frac{\partial^2}{\partial v_1\partial v_2}\log\frac{F_m(a_1)}{F_m(a_2)} = F_m(a_1)Q_m^{(2)}(a_1)$$
$$\sum_{j=1}^{m}\sum_{i=1}\frac{2y_j}{(x_i-a_1)F'_m(x_j)Q_m^{(2)}(x_j)}\frac{\partial}{\partial x_j}\frac{2y_iF_m(a_2)Q_m^{(2)}(a_1)}{F'_m(x_i)Q_m^{(2)}(x_i)(x_i-a_2)}\frac{(a_1-a_2)}{(x_i-a_1)(x_i-a_2)}.$$

The right hand side is

$$F_m(a_1)F_m(a_2)\Big(\sum_{i=1}^{m}\frac{1}{F'_m(x_i)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)(a_2-a_1)}{(x-a_1)^2(x-a_2)^2(Q_m^{(2)}(x))^2F'_m(x)}\right)\right]_{x=x_i}$$

$$-\sum_{k,l,k\neq l}\frac{2y_ky_l(a_2-a_1)}{F'_m(x_k)F'_m(x_l)(x_l-a_1)(x_l-a_2)Q_m^{(2)}(x_l)(x_k-a_1)(x_k-a_2)Q_m^{(2)}(x_k)(x_l-x_k)}\Big).$$

Then the proof of Theorem 3.1 is completely done due to next lemma. ∎

Lemma 3.2. (1)

$$\begin{aligned}
&\sum_{i=1}^{m} \frac{1}{F_m'(x_i)} \left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-a_1)^2(x-a_2)^2(Q_m^{(2)}(x)^2 F_m'(x)}\right)\right]_{x=x_i}\\
&=\frac{2}{(a_1-a_2)^2}\Big(\frac{f'(a_1)}{F_m(a_1)^2(Q_m^{(2)}(a_1))^2}+\frac{f'(a_2)}{F_m(a_2)^2(Q_m^{(2)}(a_1))^2}\\
&+\frac{f'(a_{m+1})(a_1-a_2)^2}{(a_{m+1}-a_1)(a_{m+1}-a_2)F_m(a_{m+1})^2(Q_m^{(2)\prime}(a_{m+1}))^2}\\
&+\cdots\\
&+\frac{f'(a_g)(a_1-a_2)^2}{(a_g-a_1)(a_g-a_2)F_m(a_g)^2(Q_m^{(2)\prime}(a_g))^2}\Big).
\end{aligned}$$

(2)

$$\sum_{k,l,k\neq l}\frac{2y_k y_l(a_2-a_1)}{F_m'(x_k)F_m'(x_l)(x_l-a_1)(x_l-a_2)Q_m^{(2)}(x_l)(x_k-a_1)(x_k-a_2)Q_m^{(2)}(x_k)(x_l-x_k)}\Big)=0.$$

Proof. : (1) will be proved by the following residual computations: Let ∂C_g^o be the boundary of a polygon representation C_g^o of C_g,

$$\oint_{\partial C_g^o}\frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2}dx=0. \tag{3.4}$$

The divisor of the integrand of (3.4) is

$$\sum_{i=1}^{2g+1}(b_i,0)-4\sum_{i=1,2,m+1,m+2,\cdots,g}(a_i,0)-2\sum_{i=1}^{m}(x_i,y_i)-2\sum_{i=1}^{m}(x_i,-y_i)+3\infty$$

We check these poles: First we consider the contribution around ∞ point.

$$\begin{aligned}
&\mathrm{res}_{(x_k,\pm y_k)}\frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2}dx\\
&\qquad=\frac{1}{F_m'(x_k)}\left[\frac{\partial}{\partial x}\left(\frac{f(x)}{(x-a_1)^2(x-a_2)^2(Q_m^{(2)}(x))^2F_m'(x)}\right)\right]_{x=x_k}.
\end{aligned}$$

At the point $(a_1,0)$, noting that the local parameter t is given by $t=\sqrt{(x-a_1)}$ there, we have

$$\mathrm{res}_{(a_1,0)}\frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2}dx=\frac{2f'(a_1)}{(a_1-a_2)^2F_m(a_1)^2(Q_m^{(2)}(a_1))^2}.$$

The residue at $(a_2,0)$ is similarly obtained. For the points $(a_k,0)$ $(g\geq k>m)$, we have

$$\mathrm{res}_{(a_k,0)}\frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2}dx=\frac{2f'(a_k)}{(a_k-a_1)^2(a_k-a_2)^2F_m(a_2)^2(Q_m^{(2)\prime}(a_k))^2}.$$

By arranging them, we obtain (1). (2) is obvious. □

As a corollary, we have Weierstrass relation (1.2) which was proved in [Ma]:

Corollary 3.3. *For $m = g$ case, we have the Weierstrass relation for a general curve C_g,*

$$\frac{\partial}{\partial v_r^{(g)}} \frac{\partial}{\partial v_s^{(g)}} \log \frac{\mathrm{al}_r^{(g)}}{\mathrm{al}_s^{(g)}} = \frac{1}{(a_r - a_s)} \left(f'(a_r) \left(\frac{\mathrm{al}_s^{(m)}}{\mathrm{al}_r^{(m)}} \right)^2 + f'(a_s) \left(\frac{\mathrm{al}_r^{(m)}}{\mathrm{al}_s^{(m)}} \right)^2 \right). \tag{3.5}$$

Now we will give our finial proposition as corollary.

Corollary 3.4. *For a curve satisfying the relations $c_j = a_j$ for $j = m+1, \cdots, g$, $\mathrm{al}_r^{(m)}$ and $\mathrm{al}_s^{(m)}$ $(r, s \in \{1, 2, \cdots, m\})$ over Ξ_m in (2.5) obey the relation,*

(3.6)

$$\frac{\partial}{\partial v_r^{(m)}} \frac{\partial}{\partial v_s^{(m)}} \log \frac{\mathrm{al}_r^{(m)}}{\mathrm{al}_s^{(m)}} = \frac{1}{(a_r - a_s)} \left(\frac{f'(a_r)}{(Q_m^{(2)}(a_r))^2} \left(\frac{\mathrm{al}_s^{(m)}}{\mathrm{al}_r^{(m)}} \right)^2 + \frac{f'(a_s)}{(Q_m^{(2)}(a_s))^2} \left(\frac{\mathrm{al}_r^{(m)}}{\mathrm{al}_s^{(m)}} \right)^2 \right).$$

Proof. Since the condition $c_j = a_j$ for $j = m+1, \cdots, g$ means $f'(a_j) = 0$ for $j = m+1, \cdots, g$, Theorem 3.1 reduces to this one. □

Under the same assumption of Corollary 3.4, letting $A = \dfrac{2\sqrt{f'(a_r) f'(a_s)}}{(a_r - a_s) Q_m(a_r) Q_m(a_s)}$, and

$$\phi_m^{(r,s)}(u) := \frac{1}{\sqrt{-1}} \log \sqrt{\frac{f'(a_r)}{f'(a_s} \frac{Q_m(a_r)}{Q_m(a_s)} \frac{F_m(a_r)}{F_m(a_s)}},$$

defined over Ξ_m, $\phi_m^{(r,s)}$ obeys the sin-Gordon equation,

$$\frac{\partial}{\partial v_r^{(m)}} \frac{\partial}{\partial v_s^{(m)}} \phi_m^{(r,s)} = A \cos(\phi_m^{(r,s)}). \tag{3.7}$$

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